Linear Kernelizations for Restricted 3-Hitting Set Problems

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Abstract. The 3-HITTING SET problem is also called the VERTEX COVER problem on 3-uniform hypergraphs. In this paper, we address kernelizations of the VERTEX COVER problem on 3-uniform hypergraphs. We show that this problem admits a linear kernel in three classes of 3-uniform hypergraphs. We also obtain lower and upper bounds on the kernel size for them by the parametric duality.

1 Introduction

Let C be a collection of subsets of a finite set S. A *hitting set* is a subset of S that has a nonempty intersection with every element of C. The HITTING SET problem is to decide whether there is a hitting set with cardinality at most k for a given (S, C). If the cardinality of every element of C is upper-bounded by a fixed natural number d, then the corresponding problem would be a d-HITTING SET problem. In the parameterized complexity [13,23,15], it is known that the d-HITTING SET problem is *fixed-parameter tractable*, i.e., solvable in time $O(f(k) \cdot n^c)$ for some constant c independent of k.

The 3-HITTING SET problem, a special case of the d-HITTING SET problem, is a focal point of researches in the parameterized complexity. Formally, it is defined as follows:

> 3-HITTING SET PROBLEM
>
> **Instance:** A collection C of subsets with cardinality 3 of a finite set S and a nonnegative integer k.
> **Parameter:** k.
> **Problem:** Is there a hitting set S' with $|S'| \leq k$.

Actually, (S, C) could be regarded as a hypergraph such that S and C correspond to the sets of vertices and hyperedges respectively. In this sense, the hitting set is equivalent to the vertex cover, and the 3-HITTING SET problem is the same as the VERTEX COVER problem[1] on 3-*uniform* hypergraphs in which every hyperedge consists of 3 vertices.

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[1] For simplicity, we study the 3-HITTING SET problem from the viewpoint of hypergraph theory.

A parameterized problem Q is a set of instances with the form (I, k) where I is the input instance and k is a nonnegative integer. A *kernelization* is a polynomial-time preprocessing procedure that transforms one instance (I, k) of a problem into another (I', k') of it such that

- $k' \leq k$.
- $|I'| \leq f(k')$, where f is a computable function.
- (I, k) is a "yes"-instance if and only if (I', k') is a "yes"-instance.

Hence, a kernelization can shrink the instance until that its size depends only on the parameter, so that we could find an algorithm, at least a brute force one, to efficiently solve the problem on the kernel $f(k')$. It is apparent that a problem is fixed-parameter tractable if and only if it has a kernelization.

The development of kernelizations provides a new approach for practically solving some NP-hard problems including the VERTEX COVER problem on 3-uniform hypergraphs. Presently, these kernelizations have found their way into numerous applications in many fields, e.g. bioinformatics [25,8], computer networks [19] as well as software testing [16].

1.1 Related Work

Buss [7] has given a kernelization with kernel size $O(k^2)$ for the VERTEX COVER problem on graphs by putting "high degree elements" into the cover. Similar to Buss' reduction, Niedermeier and Rossmanith [24] have proposed a cubic-size kernelization for the VERTEX COVER problem on 3-uniform hypergraphs.

Fellows *et al.* [2,12,14] have introduced the crown reduction and obtained a $3k$-kernelization for the VERTEX COVER problem on graphs. Recently, Abu-Khzam [1] has reduced further the kernel of this problem on 3-uniform hypergraphs to quadratic size by employing the crown reduction.

It is known that an optimal solution to the linear programming for the VERTEX COVER problem on graphs can be transformed into the *half-integral* form, i.e., the variables take one of three possible values $\{0, \frac{1}{2}, 1\}$. Based on this form, Nemhauser and Trotter's Theorem [22,5,2,18] guarantees the VERTEX COVER problem has a $2k$-kernelization on graphs. We also wish to follow the above idea to get a linear kernelization of the VERTEX COVER problem on 3-uniform hypergraphs. However, it seems impossible to achieve the goal because Lovász [20] and Chung *et al.* [11] have shown that the optimal solution to the linear programming has no such *half-integral* form.

Very recently, Fellows *et al.* [6] propose a new method which allows us to show that many problems do not have polynomial size kernels under reasonable complexity-theoretic assumptions.

1.2 Our Work

In this paper, we show that the VERTEX COVER problem in three classes of 3-uniform hypergraphs has a linear kernelization. Moreover, we provide lower and upper bounds on the kernel size for them by the parametric duality.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and notations. In Section 3, we study the VERTEX COVER problem on quasi-regularizable 3-uniform hypergraphs, and show that this problem has a linear kernelization. In Section 4 and 5, we also show this problem admits a linear kernelization on bounded-degree and planar 3-uniform hypergraphs respectively. Furthermore, we present lower and upper bounds on the kernel size for them. Finally, we present some questions left open and discuss the future work in Section 6.

2   Preliminaries

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with $\mathcal{V} = \{v_1, \cdots, v_n\}$ and $\mathcal{E} = \{B_1, \cdots, B_m\}$. For a hyperedge B_i, if $|B_i| = 1$, then we call it a *self-loop*. If there exists another hyperedge B_j with $B_j \subset B_i$, then we call B_i *dominated* by B_j. For a vertex v, $E(v) = \{B_i \in \mathcal{H} \mid v \in B_i\}$ and $|E(v)|$ are the *incidence set* and the *degree* of v respectively. Similarly, $E(A) = \bigcup_{v \in A} E(v)$ for a vertex subset A. If there exists another vertex w with $E(w) \subseteq E(v)$, then we call w *dominated* by v. If $w, v \in B_i$, then we call v an *adjacent* vertex of w.

Similar to the definition of regular graphs, if all the vertices of a hypergraph $\mathcal{H}$ have the same degree, then we call $\mathcal{H}$ *regular*.

Definition 1. *If the resulting hypergraph is regular after replacing each hyperedge B_i of $\mathcal{H}$ with k_i-multiple hyperedges ($k_i \geq 0$), then $\mathcal{H}$ is quasi-regularisable.*

Here, we define *k-multiple* edges B_i k copies of B_i. Note that $k = 0$ means that B_i is removed from the hypergraph.

A planar graph can be embedded on a plane without edge intersections except at the endpoints. Analogously, we can define a planar hypergraph.

Definition 2. *$G_\mathcal{H} = (V_1 \cup V_2, E)$ is a bipartite incidence graph of the hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ if $G_\mathcal{H}$ satisfies the following conditions:*

1. $V_1 = \mathcal{V}$.
2. $V_2 = \{v_B \mid B \in \mathcal{E}\}$.
3. $E = \{\{v, v_B\} \mid v \in V_1 \text{ and } v_B \in V_2 \text{ and } v \in B\}$.

Definition 3. *A hypergraph $\mathcal{H}$ is planar if $G_\mathcal{H}$ is planar.*

The optimization version of the VERTEX COVER problem is often concerned in kernelizations [2,15]. Assigning a 0-1 variable for each vertex, it is easy to establish the integer programming for the optimal version of the VERTEX COVER problem on $\mathcal{H}$ as follows:

$$\begin{aligned} &\min \sum_{i=1}^n x_i \\ \text{subject to: } &x_i + x_j + x_k \geq 1, \{v_i, v_j, v_k\} \in \mathcal{E}; \\ &x_i \in \{0, 1\}, \qquad v_i \in \mathcal{V}. \end{aligned} \qquad (1)$$

The optimal value of (1) is called the *node-covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$. Relaxing the restriction to rational number field, we can obtain the corresponding linear programming:

$$\begin{aligned}
&\min \sum_{i=1}^{n} x_i \\
&\text{subject to: } x_i + x_j + x_k \geq 1, \ \{v_i, v_j, v_k\} \in \mathcal{E}; \\
&\qquad\quad 0 \leq x_i \leq 1 \qquad v_i \in \mathcal{V}.
\end{aligned} \tag{2}$$

The optimal value of (2) is called the *fractional node-covering number* of $\mathcal{H}$, denoted by $\tau^*(\mathcal{H})$.

3 Quasi-regularizable 3-uniform Hypergraphs

In this section, we consider quasi-regularizable 3-uniform hypergraphs and show that the VERTEX COVER problem has a linear kernelization on them.

Theorem 1. *Let $\mathcal{H}$ be a quasi-regularizable 3-uniform hypergraph. The* VERTEX COVER *problem admits a problem kernel of size $3k$ on $\mathcal{H}$.*

Proof. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, \cdots, v_n\}$ and $\mathcal{E} = \{B_1, \cdots, B_m\}$. Since $\mathcal{H}$ is quasi-regularizable, we can transform it into an r-regular hypergraph $\mathcal{H}' = (\mathcal{V}, \mathcal{E}')$ with $r > 0$. If m' denotes the number of hyperedges in $\mathcal{H}'$, then $m' = \frac{1}{3}rn$.

Relaxing (1), we establish the linear programming for the VERTEX COVER problem on $\mathcal{H}'$,

$$\begin{aligned}
&\min \sum_{i=1}^{n} x_i \\
&\text{subject to: } x_i + x_j + x_k \geq 1, \ \{v_i, v_j, v_k\} \in \mathcal{E}'; \\
&\qquad\quad 0 \leq x_i \leq 1 \qquad v_i \in \mathcal{V}.
\end{aligned} \tag{3}$$

It is easy to see that $x_1 = \cdots = x_n = \frac{1}{3}$ is a feasible solution. Thus, we have

$$\tau^*(\mathcal{H}') \leq \sum_{i=1}^{n} x_i = \frac{1}{3}n. \tag{4}$$

On the other hand, let $(x_1^*, \cdots, x_n^*)$ be an optimal solution of (3). Note that the optimal solution must satisfy all the restrictions. Summing up all the inequalities in (3), it is not difficult to get

$$r \cdot \tau^*(\mathcal{H}') = r \sum_{i=1}^{n} x_i^* \geq m' = \frac{1}{3}nr \tag{5}$$

By (4) and (5), we have $\tau^*(\mathcal{H}') = \frac{1}{3}n$. If $k < \tau(\mathcal{H})$, then $\mathcal{H}$ has no vertex cover with size at most k. Therefore, we have $n \leq 3k$ because $\tau^*(\mathcal{H}) = \tau^*(\mathcal{H}') \leq \tau(\mathcal{H}) = \tau(\mathcal{H}') \leq k$. $\qquad\square$

4 Bounded-degree 3-uniform Hypergraphs

In this section, we give linear kernelizations for the VERTEX COVER problem and its dual problem on bounded-degree 3-uniform hypergraphs, respectively. Meanwhile, we derive lower bounds on kernel sizes for these two problem.

Theorem 2. *Let $\mathcal{H}$ be a 3-uniform hypergraph with bounded degree d. The* VERTEX COVER *problem admits a problem kernel of size dk on $\mathcal{H}$.*

Proof. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, \cdots, v_n\}$ and $\mathcal{E} = \{B_1, \cdots, B_m\}$. Assume that d_i is the degree of v_i. Then, we establish the integer programming (1) and the linear programming (2) for the VERTEX COVER problem on $\mathcal{H}$.

If $k < \tau(\mathcal{H})$, then $\mathcal{H}$ has no vertex cover with size at most k. Summing up all the inequalities in (2),

$$m \le \sum_{i=1}^{m} \sum_{v_j \in B_i} x_j = \sum_{i=1}^{n} d_i x_i \le d \sum_{i=1}^{n} x_i$$

Since $\tau^*(\mathcal{H}) \le \tau(\mathcal{H}) \le k$, we have $m \le d \cdot \tau^*(\mathcal{H}) \le d \cdot \tau(\mathcal{H}) \le d \cdot k$. $\qquad\square$

Chen *et al.* [9] have studied the parametric duality and kernelization. They have proposed the lower-bound technique on the kernel size.

Theorem 3 ([9]). *Let (P, s) be an NP-hard parameterized problem. Suppose that P admits an αk kernelization and its dual P_d admits an $\alpha_d k_d$ kernelization, where $\alpha, \alpha_d \ge 1$. If $(\alpha - 1)(\alpha_d - 1) < 1$, then P=NP.*

Here, $k_d = s(I) - k$ and $s : \sigma^* \times \mathbb{N} \to \mathbb{N}$ is a *size function* for a parameterized problem P if

- $0 \le k \le s(I, k)$
- $s(I, k) \le |I|$
- $s(I, k) = s(I, k')$ for all appropriate k, k'.

For the VERTEX COVER problem, it is clear that its dual problem is INDEPENDENT SET problem.

INDEPENDENT SET PROBLEM

Instance: A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and a nonnegative integer k.
Parameter: k.
Problem: Is there a vertex subset $I \subseteq \mathcal{V}$ such that $|I| \ge k$ and each hyperedge contains no two vertices of I.

Theorem 4. *Let $\mathcal{H}$ be a 3-uniform hypergraph with bounded degree d. The* INDEPENDENT SET *problem admits a problem kernel of size $(2d + 1)dk$ on $\mathcal{H}$.*

Proof. Without loss of generality, let I be a maximum independent set of $\mathcal{H}$ and $|I| \leq k$. Since $\mathcal{H}$ is a 3-uniform hypergraph with bounded degree d, the number of its adjacent vertices is bounded by $2d$ for each vertex $v \in I$, and the number of hyperedges among these adjacent vertices is bounded by $2d^2$. Therefore, we have

$$|\mathcal{E}| \leq k \cdot d + 2d^2 \cdot k = (2d+1)dk$$

$\square$

By Theorem 3, we easily get lower bounds on kernel sizes for the VERTEX COVER problem and the INDEPENDENT SET problem on 3-uniform hypergraphs with bounded degree d.

Corollary 1. *For any $\varepsilon > 0$, there is no $(\frac{2d^2+d}{2d^2+d-1} - \varepsilon)k$-kernel for the VERTEX COVER problem on 3-uniform hypergraphs with bounded degree unless P=NP.*

Corollary 2. *For any $\varepsilon > 0$, there is no $(\frac{d}{d-1} - \varepsilon)k$-kernel for the INDEPENDENT SET problem on 3-uniform hypergraphs with bounded degree unless P=NP.*

5   Planar 3-uniform Hypergraphs

In this section, we exhibit three equivalent parameterized problems on planar 3-uniform hypergraphs. Building on them, we provide a linear kernelization and a lower bound on the kernel size for the VERTEX COVER problem on planar 3-uniform hypergraphs.

Theorem 5. *Let $\mathcal{H}$ be a planar 3-uniform hypergraph. The VERTEX COVER problem admits a problem kernel of size $67k$ on $\mathcal{H}$.*

This proof consists of two parts. First, we construct two parameterized problems and show that they are equivalent to the VERTEX COVER problem on planar 3-uniform hypergraphs. In the second part, we show Theorem 5 based on the known linear kernelization [3,4,9] for the DOMINATING SET on planar graphs.

5.1   Equivalent Problems

A *dominating set* of a graph is a vertex subset D such that either $v \in D$ or v is adjacent to some vertex of D for each vertex v of the graph. $G_{\mathcal{H}}$ and $G_{\mathcal{H}}^K$ denote the bipartite incidence graph and the local complete graph of a hypergraph $\mathcal{H}$.

Definition 4. *$G_{\mathcal{H}}^K$ is the local complete graph of $\mathcal{H}$ if $G_{\mathcal{H}}^K$ satisfies the following conditions:*

1. *$V(G_{\mathcal{H}}^K) = V(G_{\mathcal{H}}) = V_1 \cup V_2$.*
2. *$E(G_{\mathcal{H}}^K) = E(G_{\mathcal{H}}) \cup \{\{x,y\} \mid \{x,y\} \subseteq B \in \mathcal{E}\}$.*

Lemma 1. *If $\mathcal{H}$ is 3-uniform and planar, then $G_{\mathcal{H}}^K$ is planar.*

Proof. It is trivial. □

Here, we consider two parameterized problems. One parameterized problem is the QUASI-DOMINATING SET problem.

QUASI-DOMINATING SET PROBLEM

Instance: A bipartite graph $G = (V_1 \cup V_2, E)$ and a nonnegative integer k.
Parameter: k.
Problem: Is there $D \subseteq V_1$ such that $\{v \mid \{v, w\} \in E \text{ and } w \in D\} = V_2$ and $|D| \leq k$?

It is apparent that the VERTEX COVER problem on $\mathcal{H}$ is equivalent to the QUASI-DOMINATING SET problem on $G_{\mathcal{H}}$.

The other is the DOMINATING SET problem.

DOMINATING SET PROBLEM

Instance: A graph $G = (V, E)$ and a nonnegative integer k.
**Parameter:** k.
Problem: Is there a dominating set $D \subseteq V$ with $|D| \leq k$?

Lemma 2. *Let $\mathcal{H}$ be a 3-uniform hypergraph. $(G_{\mathcal{H}}^K, k)$ is a YES-instance of the DOMINATING SET problem if and only if $(G_{\mathcal{H}}, k)$ is a YES-instance of the QUASI-DOMINATING SET problem.*

Proof. Let D be a dominating set of $G_{\mathcal{H}}^K$. If $D \cap V_2 = \emptyset$, then D is also a solution of the QUASI-DOMINATING SET problem on $G_{\mathcal{H}}$.

Otherwise, there is a vertex $v_B \in D \cap V_2$. By definition of $G_{\mathcal{H}}^K$, v_B corresponds to a hyperedge in $\mathcal{H}$, says $B = \{u_1, u_2, u_3\}$. Figure 1 illustrates that $D \backslash v_B \cup \{u_1\}$ is also a dominating set of $G_{\mathcal{H}}^K$.
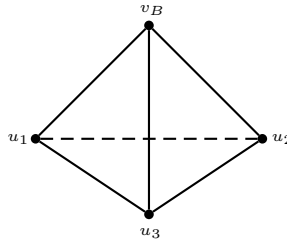
Fig. 1. v_B, u_1, u_2, u_3 in $G_{\mathcal{H}}^K$

Thus, we can find a dominating set $D' \cap V_2 = \emptyset$ of $G_{\mathcal{H}}^K$ remaining the cardinality of D. D' is a solution of the QUASI-DOMINATING SET problem on $G_{\mathcal{H}}$.

The other direction of this proof is trivial. We complete the proof. □

Corollary 3. *Let $\mathcal{H}$ be a 3-uniform hypergraph. The* VERTEX COVER *problem on $\mathcal{H}$, the* DOMINATING SET *problem on $G_{\mathcal{H}}^K$ and the* QUASI-DOMINATING SET *problem on $G_{\mathcal{H}}$ are equivalent each other.*

5.2 Kernelization

Alber *et al.* [3,4] first show that the DOMINATING SET problem on planar graphs has a problem kernel with size $335k$ based on two reduction rules. Chen *et al.* [9] extend those two reduction rules and further reduce this upper bound on the kernel size to $67k$.

Theorem 6 ([9]). *Let G be a planar graph. The* DOMINATING SET *problem admits a problem kernel of size $67k$ on G.*

All the reduction rules is founded on neighborhoods of vertices. The *neighborhood* of v in $G = (V, E)$ is $N(v) = \{x \in V \mid \{x, v\} \in E\}$. Then we can partition $N(v)$ into 3 disjoint sets as follows,

- $N_1(v) = \{u \in N(v) \mid N(u) \backslash N[v] \neq \emptyset\}$.
- $N_2(v) = \{u \in N(v) \backslash N_1(v) \mid N(u) \cap N_1(v) \neq \emptyset\}$.
- $N_3(v) = N(v) \backslash (N_1(v) \cup N_2(v))$.

Similarly, for two distinct vertices $v, w \in V$, we can also partition $N(v, w) = N(v) \cup N(w)$ into 3 disjoint sets as follows,

- $N_1(v, w) = \{u \in N(v, w) \mid N(u) \backslash N[v, w] \neq \emptyset\}$.
- $N_2(v, w) = \{u \in N(v, w) \backslash N_1(v, w) \mid N(u) \cap N_1(v, w) \neq \emptyset\}$.
- $N_3(v, w) = N(v, w) \backslash (N_1(v, w) \cup N_2(v, w))$.

As stated in [9], all the vertices of G are colored black initially. Reduction rules will color some vertices white, which means that these white vertices are excluded from an optimal dominating set of G. We repeat applying eight reduction rules in [9] to reduce the planar graph until the resulting graph, i.e. a *reduced graph*, is unchanged.

Rule 1 *If there is a black vertex $x \in N_2(v) \cup N_3(v)$ for each black vertex v, then color x white.*

As Figure 1 exhibits, there is always a black vertex $u \in V_1$ with $v \in N_2(u) \cup N_3(u)$ for any black vertex $v \in V_2$. Thus for the initial graph $G_{\mathcal{H}}^K$, this rule must color all the vertices of V_2 white. It implies that the optimal dominating set D of $G_{\mathcal{H}}^K$ contains no vertex in V_2. In other words, $D \subseteq V_1$ is a cover of $\mathcal{H}$, which means that those vertices colored white in V_1 can be removed from $\mathcal{H}$.

Note that the rest rules either color vertices white or remove some white vertices except the following two ones.

Rule 2 *If $N_3(v) \neq \emptyset$ for some black vertex v, then*

- *Remove the vertices in $N_2(v) \cup N_3(v)$ from the current graph.*
- *Add a new white vertex v' and an edge $\{v, v'\}$.*

Rule 3 *If $N_3(v, w) \neq \emptyset$ for two black vertices v, w and if $N_3(v, w)$ cannot be dominated by a single vertex from $N_2(v, w) \cup N_3(v, w)$, then*

Case 1: *If $N_3(v, w)$ can be dominated by a single vertex from $\{v, w\}$:*
- *If $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \subseteq N(w)$:*
  - *Remove the vertices in $N_3(v, w) \cup (N_2(v, w) \cap N(v) \cap N(w))$ from the current graph.*
  - *Add two new white vertices z, z' and four edges $\{v, z\}$, $\{w, z\}$, $\{v, z'\}$, $\{w, z'\}$.*
- *If $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \nsubseteq N(w)$:*
  - *Remove the vertices in $N_3(v, w) \cup (N_2(v, w) \cap N(v))$ from the current graph.*
  - *Add a new white vertex v' and an edge $\{v, v'\}$.*
- *If $N_3(v, w) \nsubseteq N(v)$ and $N_3(v, w) \subseteq N(w)$:*
  - *Remove the vertices in $N_3(v, w) \cup (N_2(v, w) \cap N(w))$ from the current graph.*
  - *Add a new white vertex w' and an edge $\{w, w'\}$.*

Case 2: *If $N_3(v, w)$ can not be dominated by a single vertex from $\{v, w\}$:*
- *Remove the vertices in $N_3(v, w) \cup N_2(v, w)$ from the current graph.*
- *Add two new white vertices v', w' and two edges $\{v, v'\}$, $\{w, w'\}$.*

It is obvious that both Rule 2 and Rule 3 add new white vertices to the graph. We categorize those new white vertices into V_2. In this sense, they are viewed as self-loops in $\mathcal{H}$.

Therefore, we achieve a kernelization for the VERTEX COVER problem on $\mathcal{H}$.

1. Construct $G_{\mathcal{H}}$ and $G_{\mathcal{H}}^K$.
2. Reduce $G_{\mathcal{H}}^K$ to a reduced graph G' by using reduction rules in [9].
3. Remove white vertices in V_1 from G'.
4. Remove all the edges between two vertices of V_1 from G'.
5. Construct a reduced hypergraph $\mathcal{H}'$ based on the resulting bipartite graph.

By Theorem 6, the proof of Theorem 5 is complete.

5.3 Dual Problem

In this section, we give a linear kernelization for the INDEPENDENT SET problem on a planar 3-uniform hypergraphs.

Theorem 7. *Let $\mathcal{H}$ be a planar 3-uniform hypergraph. The INDEPENDENT SET problem admits a problem kernel of size $40k$ on $\mathcal{H}$.*

To prove the above theorem, we first need to introduce the INDUCED MATCHING problem which is known to be $W[1]$-hard [17,21].

A *matching* M in a graph $G = (V, E)$ is a subset of edges no two of which have a common endpoint. If no two edges of M are joined by an edge of G, then M is an *induced matching*.

INDUCED MATCHING PROBLEM

Instance: A graph $G = (V, E)$ and a nonnegative integer k.
Parameter: k.
Problem: Is there an induced matching $M \subseteq E$ with $|E| \leq k$?

Theorem 8 ([17]). *Let G be a planar graph. The INDUCED MATCHING problem admits a problem kernel of size $40k$ on G.*

Since $\mathcal{H}$ is a planar 3-uniform hypergraph, $G_{\mathcal{H}}^K$ must be planar. Hence, the rest task is to show the following lemma.

Lemma 3. *The INDEPENDENT SET problem on a planar 3-uniform hypergraph $\mathcal{H}$ is equivalent to the INDUCED MATCHING problem on $G_{\mathcal{H}}^K$.*

Proof. We assign a vertex v_B in $G_{\mathcal{H}}^K$ for a hyperedge $B \in \mathcal{H}$.

($\Leftarrow$) Let M be an induced matching of $G_{\mathcal{H}}$. Without loss of generality, let $I = \{u_{i1} \mid \{u_{i1}, v_{B_i}\} \in M\}$. We claim that I is an independent set of $\mathcal{H}$. Otherwise, $\{u_{i1}, u_{j1}\} \subseteq B$ for some hyperedge B with $i \neq j$. By definition of $G_{\mathcal{H}}^K$, $\{u_{i1}, v_{B_i}\}$ and $\{u_{j1}, v_{B_j}\}$ are joined by $\{u_{i1}, u_{j1}\}$ in $G_{\mathcal{H}}^K$. It is a contradiction.

($\Rightarrow$) Without loss of generality, let I be an independent set of $\mathcal{H}$. For $u_{i1}, u_{j1} \in I$, we can find two distinct hyperedges B_i and B_j such that $u_{i1} \in B_i$ and $u_{j1} \in B_j$ by definition of independent set. It is easy to see that $\{u_{i1}, v_{B_i}\}$ and $\{u_{j1}, v_{B_j}\}$ are not joined by an edge in $G_{\mathcal{H}}^K$. Thus, $M = \{u_{i1}, v_{B_i}\} \mid u_i \in I\}$ is an induced matching of $\mathcal{H}$.

We complete this proof. $\square$

By Theorem 3, we obtain lower bounds on kernel sizes for the VERTEX COVER problem and the INDEPENDENT SET problem on planar 3-uniform hypergraphs.

Corollary 4. *For any $\varepsilon > 0$, there is no $(\frac{40}{39} - \varepsilon)k$-kernel for the VERTEX COVER problem on 3-uniform hypergraphs with bounded degree unless P=NP.*

Corollary 5. *For any $\varepsilon > 0$, there is no $(\frac{67}{66} - \varepsilon)k$-kernel for the INDEPENDENT SET problem on 3-uniform hypergraphs with bounded degree unless P=NP.*

6 Conclusion

In this paper we study kernelizations of the VERTEX COVER problem on 3-uniform hypergraphs, and show that this problem in three classes of 3-uniform hypergraphs has linear kernels. We give lower and upper bounds on the kernel size for them by the parametric duality. An interesting open question is how to decide these 3-uniform hypergraphs efficiently. It is a challenge to explore more 3-uniform hypergraphs with less linear kernels in the future. Additionally, designing better kernelization algorithms and pursuing less kernel are also challenging tasks for the VERTEX COVER problem on 3-uniform hypergraphs.

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